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Principal chiral models on non-semisimple groups

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Abstract
Generalized principal models on non-semisimple groups are defined. An ansatz for the Lax form of the equations of motion is chosen and models on two- and three-dimensional non-semisimple groups that admit this Lax formulation are classified. Only one of these models has truly nonlinear equations of motion, and the Lax pair is explicitly given. The equations of motion of all the other models can be brought to linear partial differential equations.

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1. Introduction

Integrable models in two dimensions are important theoretical laboratories for investigating possible phenomena of nonlinear theories in higher dimensions. Principal chiral models are examples of nonlinear relativistic field theories on the group manifolds. It is well known that they are classically integrable in 1 + 1 dimensions (see [1]).

Until now, the principal models were investigated mainly on semisimple groups because the bilinear forms used for the construction of the field actions were actually taken as the ad-invariant Killing metrics on the corresponding Lie algebras. As the forms should be non-degenerate, these models were defined on semisimple groups only. A few years ago Sochen [2] suggested a generalization of the principal models for metrics that are not ad-invariant (see also [3] and [4] for a different point of view). It opened up the possibility of defining the principal models on non-semisimple groups as well. An example of such model, including the Lax formulation of equations of motion was formulated in [5] but the parameters in the Lax pair could be transformed off, so that there remained no free spectral parameters, necessary for the inverse spectral method. We use a more general ansatz for the Lax operators in this paper that enables us to introduce such a free parameter.

The main topic of this paper is classification of models on the two- and three-dimensional non-semisimple groups that admit Lax formulation of a form given below.

2. Generalized principal models

Generalized principal chiral models [2] are given by the action

$$I[g] = \int d^2x\, \eta^{\mu\nu} L_{ab}(g) J_\mu^a J_\nu^b \tag{1}$$

where G is a Lie group, $\mathcal{L}(G)$ its Lie algebra,

$$J_\mu = \left(g^{-1}\partial_\mu g\right) \in \mathcal{L}(G) \tag{2}$$

g: $\mathbf{R^2} \to G, \mu, \nu \in \{0, 1\}, \eta = \mathrm{diag}(1, -1)$, L is a G-dependent symmetric non-degenerate bilinear form. We consider the bilinear form L as a metric on the group manifold and the generalization of principal models from ad-invariant Killing form on $\mathcal{L}(G)$ to more general case enables us to introduce the principal models on non-semisimple groups also.

Lie products of elements on the basis of $\mathcal{L}(G)$ define the structure coefficients

$$[t_a, t_b] = c_{ab}{}^c t_c \tag{3}$$

and on the same basis we define the coordinates of the field J_ν

$$J_\nu = g^{-1}\partial_\nu g = J_\nu^b t_b. \tag{4}$$

Fields automatically satisfy the Bianchi identities

$$\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0. \tag{5}$$

Varying the action (1) we obtain the equations of motion for the generalized principal chiral models

$$\partial_\mu J^{\mu,a} + \Gamma^a_{bc} J_\mu^b J^{\mu,c} = 0 \tag{6}$$

where the connection Γ is a sum of two parts

$$\Gamma^a_{bc} = S^a_{bc} + \gamma^a_{bc}. \tag{7}$$

S^a_{bc} is a so-called flat connection

$$S^a_{bc} = \frac{1}{2}\left(C^a_{bc} + C^a_{cb}\right) \qquad C^a_{bc} = (L^{-1})^{ap} c^q_{pb} L_{qc} \tag{8}$$

and γ^a_{bc} are Christoffel symbols for the metric $L_{ab}(g)$

$$\gamma^a_{bc} = \frac{1}{2}(L^{-1})^{ad}(U_b L_{cd} + U_c L_{bd} - U_d L_{bc}). \tag{9}$$

The vector fields U_a are defined in the local group coordinates θ_i as

$$U_a = U_a^i(\theta)\frac{\partial}{\partial\theta_i} \tag{10}$$

where the matrix U is the inverse of the matrix V of vielbein coordinates

$$U_a^i(\theta) = (V^{-1})_a^i(\theta) \qquad V_i^a = \left(g^{-1}\frac{\partial g}{\partial\theta_i}\right)^a. \tag{11}$$

Note that the connection (7) is symmetric in the lower indices

$$\Gamma^a_{bc} = \Gamma^a_{cb}. \tag{12}$$

2.1. Lax pairs

It is evident from (6) and (2) that the equations of motion of generalized principal chiral models may form highly nonlinear systems of PDEs. One of the most powerful method for solving nonlinear PDEs is the so-called inverse scattering method that transforms the PDEs to solvable system of ODEs. The inverse transform requires solving the Riemann–Hilbert problem of determining a complex function from their values at a curve (for a detailed explanation, see, e.g., [6]).

The first step of the method consists in writing the system of PDEs in terms of a commutator of two differential operators X_0, X_1 containing a free parameter that is later used as the independent variable in the associated Riemann–Hilbert problem. These operators are called Lax pair and serve to define an associated linear spectral problem defining the (direct) transform. Finding such a Lax pair for a given system of PDEs is a rather nontrivial problem.

The ansatz that we are going to use for the Lax operators X_0, X_1 of the generalized principal chiral models is

$$X_0 = \partial_0 + P_{ab} J_0^b t_a + Q_{ab} J_1^b t_a + A_a t_a \tag{13}$$

$$X_1 = \partial_1 + \tilde{Q}_{ab} J_0^b t_a + \tilde{P}_{ab} J_1^b t_a + B_a t_a \tag{14}$$

where $P, Q, \tilde{P}, \tilde{Q}$ are four arbitrary constant $\dim G \times \dim G$ matrices and A, B are two arbitrary constant vectors.

By explicit evaluation of the zero curvature condition

$$[X_0, X_1] = 0 \tag{15}$$

using the equations of motion (6) and Bianchi identities (5), and equating the coefficients of different powers and derivatives of J_μ^a, one finds the following necessary conditions that the operators X_0, X_1 must satisfy in order to form a Lax pair:

$$\tilde{P} = P, \tilde{Q} = Q \tag{16}$$

$$(P_{bp} P_{cq} - Q_{bp} Q_{cq}) c_{bc}^a = P_{ab} c_{pq}^b \tag{17}$$

$$\tfrac{1}{2} c_{cd}^a (P_{cp} Q_{dq} + P_{cq} Q_{dp}) = Q_{ab} \Gamma_{pq}^b \tag{18}$$

$$c_{cd}^a (P_{cp} B_d + A_c Q_{dp}) = 0 \tag{19}$$

$$c_{cd}^a (Q_{cp} B_d + A_c P_{dp}) = 0 \tag{20}$$

$$c_{cd}^a A_c B_d = 0. \tag{21}$$

Equation (16) is the reason for originally counterintuitive notation in equations (13) and (14). In the following we always immediately replace $\tilde{P}$ by P and $\tilde{Q}$ by Q.

In order to guarantee the equivalence between equation (15) and the equations of motion (6) one needs further restrictions on P, Q, A, B (otherwise consider e.g. $P = Q = A = B = 0$). Such a condition can be found quite easily by rewriting the left-hand side of the equation (15) and using equations (16)–(21) and the Bianchi identities (5), one gets

$$[X_0, X_1] = Q_{ab} \left(\partial_\mu J^{\mu,b} + \Gamma_{pq}^b J_\mu^p J^{\mu,q} \right). \tag{22}$$

It is now clear that equation (15) is equivalent to the equations of motion (6) if and only if the matrix Q is invertible.

To sum up, the Lax formulation (13)–(15) *is equivalent to the equations of motion if and only if the equations* (16)–(21) *hold and Q is invertible*.

Moreover, the previous equations impose a condition on Γ. Hence, we can express (18) in an equivalent form

$$\frac{1}{2} (Q^{-1})^{ba} c_{cd}^a (P_{cp} Q_{dq} + P_{cq} Q_{dp}) = \Gamma_{pq}^b \tag{23}$$

and conclude that *only the generalized principle models with the constant connection* Γ *admit the Lax formulation* (13)–(15) because the left-hand side of the previous equation is constant.

3. Abelian groups

The case of the principal models on Abelian groups having the Lax formulation (15) can be investigated rather quickly by the following method. Because in this case $c_{ab}^{c} = 0$, equation (17) is satisfied identically and equation (18) simplifies to $0 = Q_{ab}\Gamma^{b}_{pq}$. This equation represents for any given pair p, q a set of $\dim G$ linear equations for $\dim G$ variables Γ^{b}_{pq} with an invertible matrix of coefficients ($=Q$); therefore only the trivial solution $\Gamma^{b}_{pq} = 0$ is possible, leading to the model

$$\partial_\mu J^{\mu,a} = 0. \tag{24}$$

Because we may choose coordinates $\theta\colon g(\theta) = \exp\left(\sum_{i=1}^{n}\theta_i t_i\right)$, and the corresponding expression for the fields is $J^{a}_{\mu} = \partial_\mu\theta_a$, we have in such coordinates a free model

$$\partial_\mu\partial^\mu\theta_i = 0. \tag{25}$$

In the following we shall systematically explore the generalized principal models on non-semisimple two- and three-dimensional Lie groups.

4. Two-dimensional solvable group

Every non-Abelian two-dimensional connected Lie group is isomorphic to the group of affine transformations of real line. Let us denote it by $Af(1)$. This group can be conveniently realized as a matrix group consisting of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad a > 0. \tag{26}$$

A suitable parametrization of this group is

$$g(\theta_1,\theta_2) = \begin{pmatrix} \exp(\theta_1) & \theta_2 \\ 0 & 1 \end{pmatrix} \tag{27}$$

where $\theta_1, \theta_2 \in \mathbf{R}$. The basis of the corresponding Lie algebra can be chosen from

$$t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{28}$$

The nonzero structure coefficients for this choice of basis are

$$c^2_{12} = 1 \qquad c^2_{21} = -1. \tag{29}$$

The coordinates of vector fields J_μ in this basis are $(\partial_\mu\theta_1, \mathrm{e}^{-\theta_1}\partial_\mu\theta_2)$. The differential operators U_a in this case are

$$U_1 = \frac{\partial}{\partial\theta_1} \qquad U_2 = \mathrm{e}^{\theta_1}\frac{\partial}{\partial\theta_2}. \tag{30}$$

The equations of motion are

$$0 = \frac{\partial^2\theta_1}{\partial x_0^2} - \frac{\partial^2\theta_1}{\partial x_1^2} + \Gamma^1_{11}\left(\left(\frac{\partial\theta_1}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_1}{\partial x_1}\right)^2\right) + 2\Gamma^1_{12}\mathrm{e}^{-\theta_1}\left(\frac{\partial\theta_1}{\partial x_0}\frac{\partial\theta_2}{\partial x_0} - \frac{\partial\theta_1}{\partial x_1}\frac{\partial\theta_2}{\partial x_1}\right)$$
$$+ \Gamma^1_{22}\mathrm{e}^{-2\theta_1}\left(\left(\frac{\partial\theta_2}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_2}{\partial x_1}\right)^2\right) \tag{31}$$

$$0 = \mathrm{e}^{-\theta_1}\left(\frac{\partial^2\theta_2}{\partial x_0^2} - \frac{\partial^2\theta_2}{\partial x_1^2} - \frac{\partial\theta_1}{\partial x_0}\frac{\partial\theta_2}{\partial x_0} + \frac{\partial\theta_1}{\partial x_1}\frac{\partial\theta_2}{\partial x_1}\right) + \Gamma^2_{11}\left(\left(\frac{\partial\theta_1}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_1}{\partial x_1}\right)^2\right) + 2\Gamma^2_{12}\mathrm{e}^{-\theta_1}$$
$$\times\left(\frac{\partial\theta_1}{\partial x_0}\frac{\partial\theta_2}{\partial x_0} - \frac{\partial\theta_1}{\partial x_1}\frac{\partial\theta_2}{\partial x_1}\right) + \Gamma^2_{22}\mathrm{e}^{-2\theta_1}\left(\left(\frac{\partial\theta_2}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_2}{\partial x_1}\right)^2\right). \qquad (32)$$

These equations were already investigated in [5] but only for diagonal metric L and less general ansatz of Lax operators.

To find models admitting Lax formulation one can proceed in the following way. Equation (17) is equivalent to two equations

$$P_{12} = 0 \qquad P_{22}(P_{11} - 1) = Q_{11}Q_{22} - Q_{21}Q_{12}\ (=\det\ Q). \qquad (33)$$

Because the matrix Q is invertible, i.e. $\det\ Q \neq 0$, the second condition can be rewritten

$$P_{22} = \frac{\det\ Q}{P_{11} - 1}. \qquad (34)$$

Equation (18) should be considered first for $a = 1$. Then the left-hand side of (18) vanishes and one finds

$$0 = Q_{11}\Gamma^1_{pq} + Q_{12}\Gamma^2_{pq} \qquad \forall p, q. \qquad (35)$$

We divide our investigation into two cases depending on the value of Q_{12}.

4.1. Case $Q_{12} \neq 0$

In this case we immediately find that $\Gamma^2_{pq} = K\Gamma^1_{pq}, \forall p, q$ (where $K = -Q_{11}/Q_{12}$) and the defining equations (7) for Γ can be rewritten in an equivalent form

$$\frac{\partial L_{11}}{\partial\theta_1} = 2\Gamma^1_{11}(L_{11} + KL_{12}) \qquad (36)$$
$$\frac{\partial L_{11}}{\partial\theta_2} = \mathrm{e}^{-\theta 1}\left(2\Gamma^1_{12}L_{11} + 2K\Gamma^1_{12}L_{12} - L_{12}\right) \qquad (37)$$
$$\frac{\partial L_{12}}{\partial\theta_1} = \Gamma^1_{12}L_{11} + K\Gamma^1_{11}L_{22} + \Gamma^1_{11}L_{11} + K\Gamma^1_{11}L_{12} + \frac{1}{2}L_{12} \qquad (38)$$
$$\frac{\partial L_{12}}{\partial\theta_2} = \frac{\mathrm{e}^{-\theta 1}}{2}\left(2\Gamma^1_{22}L_{11} + 2\Gamma^1_{12}(KL_{22} + L_{12}) + 2K\Gamma^1_{22}L_{12} - L_{22}\right) \qquad (39)$$
$$\frac{\partial L_{22}}{\partial\theta_1} = 2\Gamma^1_{12}L_{12} + L_{22} + 2K\Gamma^1_{12}L_{22} \qquad (40)$$
$$\frac{\partial L_{22}}{\partial\theta_2} = 2\mathrm{e}^{-\theta_1}\Gamma^1_{22}(L_{12} + KL_{22}) \qquad (41)$$

where K and Γ's are constants. From these equations it is rather easy to calculate following necessary conditions for the existence of the metric L. Using the equations (36)–(41) one evaluates the difference of second derivatives

$$\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}L_{ij} - \frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_1}L_{ij} \qquad (42)$$

in terms of L_{kl}. Since this difference must be zero, one obtains a set of three (for $(i, j) = (1, 1), (1, 2), (2, 2)$) linear equations for L_{kl}. In order to have a nontrivial (nonzero) solution, the matrix of this set of equations must have a zero determinant, i.e.,

$$-\frac{1}{2}\mathrm{e}^{-3\theta_1}\left(K\Gamma^1_{22} + \Gamma^1_{12}\right)\left[4K^2\Gamma^1_{11}\left(\Gamma^1_{22}\right)^2 - 4K^2\left(\Gamma^1_{12}\right)^2\Gamma^1_{22} + 8K\Gamma^1_{11}\Gamma^1_{12}\Gamma^1_{22} - 8K\left(\Gamma^1_{12}\right)^2\right.$$
$$\left. - 3\Gamma^1_{22} + 4\left(\Gamma^1_{11}\right)^2\Gamma^1_{22} - \Gamma^1_{11}\Gamma^1_{22} - 4\Gamma^1_{11}\left(\Gamma^1_{12}\right)^2 + 4\left(\Gamma^1_{12}\right)^2\right] = 0.$$

As this must hold for all θ_1, we get either

$$\Gamma_{12}^1 = -K\Gamma_{22}^1 \tag{43}$$

or

$$4K^2\Gamma_{11}^1\left(\Gamma_{22}^1\right)^2 - 4K^2\left(\Gamma_{12}^1\right)^2\Gamma_{22}^1 + 8K\Gamma_{11}^1\Gamma_{12}^1\Gamma_{22}^1 - 8K\left(\Gamma_{12}^1\right)^2 - 3\Gamma_{22}^1 + 4\left(\Gamma_{11}^1\right)^2\Gamma_{22}^1 - \Gamma_{11}^1\Gamma_{22}^1 - 4\Gamma_{11}^1\left(\Gamma_{12}^1\right)^2 + 4\left(\Gamma_{12}^1\right)^2 = 0. \tag{44}$$

It can be found by careful investigation that if (44) holds, then the only possible metrics L are singular (for all values of θ_1, θ_2).

The remaining possibility is that (43) holds. In this case, from (42) one obtains further conditions on L's and Γ's, namely that either

$$\Gamma_{11}^1 = -\frac{1}{2} \qquad \Gamma_{12}^1 = \Gamma_{22}^1 = 0 \tag{45}$$

or

$$L_{12} = -KL_{22}, L_{11} = \frac{\left(-2K^2\Gamma_{11}^1\Gamma_{22}^1 + 2\Gamma_{11}^1 + 4K^4\Gamma_{22}^1 + 1 + 4K^2\Gamma_{22}^1\right)L_{22}}{2\Gamma_{22}^1\left(2K^2\Gamma_{22}^1 - 2\Gamma_{11}^1 + 3\right)}. \tag{46}$$

Otherwise the metric L is singular.

In case (45) one can compute the metric L and the connection Γ from the equations (36)–(41) and the matrices P, Q from equations (18)–(21), but the resulting matrix Q is not invertible, i.e. the Lax formulation (15) is not equivalent to the equations of motion (6).

In case (46) we can again solve equations (36)–(41). The resulting connection and the metric are

$$\Gamma_{11}^1 = \frac{1}{2} + K^2\Gamma_{22}^1 \qquad \Gamma_{22}^1 = -K\Gamma_{22}^1 \tag{47}$$

$$L_{11} = \frac{\Gamma_{11}^1}{\Gamma_{22}^1}\alpha e^{\theta_1} \qquad L_{12} = -K\,\alpha e^{\theta_1} \qquad L_{22} = \alpha e^{\theta_1} \tag{48}$$

where $\Gamma_{22}^1, \alpha \in \mathbf{R}\backslash\{0\},\ K \in \mathbf{R}$ are parameters of the model. In the following we will denote $\Gamma_{22}^1 = -\kappa^2/2$.

The resulting equations of motion can be found substituting the above given Γ into equations (31) and (32). To get the Lax operators for this model we still have to solve equations (18)–(21). The solutions in this case depend on three arbitrary parameters λ, ρ, σ where $\sigma \neq 0$

$$P = \begin{pmatrix} \frac{1}{2} & 0 \\ \epsilon_1(\kappa K\sigma + \rho) & -\epsilon_1\sigma\kappa \end{pmatrix} \tag{49}$$

$$Q = \begin{pmatrix} \frac{\epsilon_1 K}{2}\kappa & -\frac{\epsilon_1}{2}\kappa \\ \sigma + K\kappa\rho & -\kappa\rho \end{pmatrix} \qquad \epsilon_1 = \pm 1 \tag{50}$$

$$A = (\lambda, 2\lambda(\epsilon_1\rho - \epsilon_2\sigma)) \tag{51}$$

$$B = (\epsilon_2\lambda, 2\lambda(\epsilon_1\epsilon_2\rho - \sigma)) \qquad \epsilon_2 = \pm 1. \tag{52}$$

The Lax operator X_0 then reads (for simplicity we set $\epsilon_1 = \epsilon_2 = +1$)

$$X_0 = \begin{pmatrix} \partial_0 + \frac{1}{2}Y_0 + \lambda, & \sigma Y_1 + \rho Y_0 + 2\lambda(\rho - \sigma) \\ 0 & \partial_0 \end{pmatrix} \tag{53}$$

where Y_μ are linear functions of the fields J_μ^a,

$$Y_0 = J_0^1 - \kappa J_1^2 + K\kappa J_1^1 \tag{54}$$

$$Y_1 = J_1^1 - \kappa J_0^2 + K\kappa J_0^1. \tag{55}$$

The expression for X_1 can be obtained from (53) by an interchange of indices 0, 1 in (53).

We can transform the Lax operators to the form with one parameter only by the similarity transform $\tilde{X}_\mu = T X_\mu T^{-1}$ with

$$T = \begin{pmatrix} 1 & 2\rho \\ 0 & \sigma \end{pmatrix}. \tag{56}$$

The transformed Lax operator then is of the form

$$\tilde{X}_0 = \begin{pmatrix} \partial_0 + \frac{1}{2}Y_0 + \lambda, & Y_1 - 2\lambda \\ 0 & \partial_0 \end{pmatrix}. \tag{57}$$

An analogous expression is obtained for $\tilde{X}_1$.

From the formula (57) it is clear that if the ansatz (13) and (14) is chosen without the constant terms $A_a t_a$, $B_a t_a$ (cf. [5]) then $\lambda = 0$ and there is no free parameter for the inverse scattering method.

4.2. *Case $Q_{12} = 0$*

If $Q_{12} = 0$, then the invertibility of Q leads to $Q_{11} \neq 0$ and equation (35) simplifies to

$$0 = \Gamma^1_{pq}. \tag{58}$$

It is clear that equation (31) is then just the wave equation

$$\frac{\partial^2\theta_1}{\partial x_0^2} - \frac{\partial^2\theta_1}{\partial x_1^2} = 0. \tag{59}$$

Using the same approach as before for $Q_{12} \neq 0$, one finds that the defining relation of Γ (7) can be reformulated in the following way

$$\frac{\partial L_{11}}{\partial\theta_1} = 2\Gamma^2_{11}L_{12} \tag{60}$$

$$\frac{\partial L_{11}}{\partial\theta_2} = \mathrm{e}^{-\theta_1}\left(-1 + 2\Gamma^2_{12}\right)L_{12} \tag{61}$$

$$\frac{\partial L_{12}}{\partial\theta_1} = \frac{1}{2}L_{12} + \Gamma^2_{11}L_{22} + \Gamma^2_{12}L_{12} \tag{62}$$

$$\frac{\partial L_{12}}{\partial\theta_2} = \frac{1}{2}\mathrm{e}^{-\theta_1}\left(2\Gamma^2_{22}L_{12} + 2\Gamma^2_{12}L_{22} - L_{22}\right) \tag{63}$$

$$\frac{\partial L_{22}}{\partial\theta_1} = \left(1 + 2\Gamma^2_{12}\right)L_{22} \tag{64}$$

$$\frac{\partial L_{22}}{\partial\theta_2} = 2\mathrm{e}^{-\theta_1}\Gamma^2_{22}L_{22}. \tag{65}$$

The interchangeability of the ordering of partial derivatives of θ_i (equation (42)) leads to conditions

$$\Gamma^2_{12} = -\frac{1}{2} \qquad \Gamma^2_{22} = 0. \tag{66}$$

By solving (60)–(65) one finds all possible metrics in the form

$$L_{11} = \mathrm{e}^{2\theta_1}\left(\Gamma^2_{11}\right)^2\alpha + 2\mathrm{e}^{\theta_1}\Gamma^2_{11}\beta + \gamma \tag{67}$$

$$L_{12} = \Gamma^2_{11}\alpha\mathrm{e}^{2\theta_1} + \beta\mathrm{e}^{\theta_1} \qquad L_{22} = \alpha\mathrm{e}^{2\theta_1}$$

where $\alpha, \beta, \gamma, \Gamma^2_{11} \in \mathbf{R}$ are parameters such that det $L \neq 0$. By evaluation of Γ we arrive at the explicit form of equations of motion

$$\frac{\partial^2\theta_1}{\partial x_0^2} - \frac{\partial^2\theta_1}{\partial x_1^2} = 0 \qquad \mathrm{e}^{-\theta_1}\left(\frac{\partial^2\theta_2}{\partial x_0^2} - \frac{\partial^2\theta_2}{\partial x_1^2}\right) + \Gamma^2_{11}\left(\left(\frac{\partial\theta_1}{\partial x_0}\right)^2 - \left(\frac{\partial\theta_1}{\partial x_1}\right)^2\right) = 0. \tag{68}$$

The equations of motion are in fact two coupled linear wave equations; the first one is homogeneous, i.e. explicitly solvable ($\theta_1 = F(x_0 - x_1) + G(x_0 + x_1)$), and the second one contains nonlinear terms in already known θ_1 only, it is therefore just the inhomogeneous wave equation. That is why the application of the inverse spectral method is questionable in this case.

5. Three-dimensional solvable Lie groups

Structure of all three-dimensional solvable real Lie algebras can be written in the following form (see e.g. [7]):

$$\begin{aligned} [t_2, t_3] &= b_{11}t_1 + b_{12}t_2 \\ [t_3, t_1] &= b_{21}t_1 + b_{22}t_2 \\ [t_1, t_2] &= 0 \end{aligned} \tag{69}$$

where the 2×2 matrix $B = (b_{ij})$ is one of the following

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 1 \\ -1 & \sigma \end{pmatrix}, \quad \begin{pmatrix} \sigma & 1 \\ -1 & -\sigma \end{pmatrix}$$

σ is a positive real number. Each of these algebras can be realized as a matrix algebra in the form

$$\left\{ \begin{pmatrix} (1 + b_{21})z & -b_{11}z & x \\ b_{22}z & (1 - b_{12})z & y \\ 0 & 0 & z \end{pmatrix} \middle| \, x, y, z \in \mathbf{R} \right\}. \tag{70}$$

For convenience, in the following we will denote by capital letters indices going from 1 to 2 only (e.g. $A \in \{1, 2\}$), other index conventions remain unchanged.

The structure coefficients are

$$c_{12}{}^{q} = 0 \qquad c_{A3}{}^{3} = 0 \tag{71}$$
$$c_{31}{}^{1} = b_{21} \qquad c_{31}{}^{2} = b_{22} \qquad c_{23}{}^{1} = b_{11} \qquad c_{23}{}^{2} = b_{12}. \tag{72}$$

Considering the equations (17) for $a = 3$ one finds $P_{3B}c_{pq}{}^{B} = 0$, i.e. for $(p, q) = (3, 1)$ and $(p, q) = (2, 3)$

$$P_{31}b_{21} + P_{32}b_{22} = 0 \qquad P_{31}b_{11} + P_{32}b_{12} = 0. \tag{73}$$

We divide our investigation into two possibilities, first det $B \neq 0$ ($\Rightarrow P_{31} = P_{32} = 0$) and second det $B = 0$.

5.1. Three–dimensional solvable groups with det $B \neq 0$

Considering the case of $\det B \neq 0$ (i.e. all cases in the previous classification except the Abelian algebra ($B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$), the nilpotent Heisenberg algebra ($B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$) and the case $B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$), we can conclude that

$$P_{31} = P_{32} = 0. \tag{74}$$

Furthermore, considering equations (17) for $(p, q) = (1, 2)$ one finds

$$c^A_{13}(Q_{31}Q_{12} - Q_{11}Q_{32}) + c^A_{23}(Q_{31}Q_{22} - Q_{21}Q_{32}) = 0. \tag{75}$$

Using the relations between the structure constants and the matrix B, one finds that the linear set of equations (75) for $Q_{31}Q_{12} - Q_{11}Q_{32}$ and $Q_{31}Q_{22} - Q_{21}Q_{32}$ has only trivial solution (for $\det B \neq 0$), i.e.

$$Q_{11}Q_{32} = Q_{31}Q_{12} \qquad Q_{21}Q_{32} = Q_{31}Q_{22}. \tag{76}$$

It follows that $\det Q = Q_{33}(Q_{11}Q_{22} - Q_{12}Q_{21})$ and if $Q_{31} \neq 0$ or $Q_{32} \neq 0$ then $Q_{11}Q_{22} = Q_{21}Q_{12}$ and $\det Q = 0$.

We are therefore led to

$$Q_{31} = Q_{32} = 0. \tag{77}$$

Using equations (18) one finds for $a = 3$ that

$$0 = Q_{3b}\Gamma^b_{pq} = Q_{33}\Gamma^3_{pq} \qquad \text{i.e.} \quad \Gamma^3_{pq} = 0 \tag{78}$$

and for $p = A, q = B$

$$0 = \frac{1}{2}c^a_{CD}(P_{CA}Q_{DB} + P_{CB}Q_{DA}) = Q_{ab}\Gamma^b_{AB} \tag{79}$$

leading together with (78) to

$$\begin{aligned} 0 &= Q_{11}\Gamma^1_{AB} + Q_{12}\Gamma^2_{AB} \\ 0 &= Q_{21}\Gamma^1_{AB} + Q_{22}\Gamma^2_{AB}. \end{aligned} \tag{80}$$

Since $\det Q = Q_{33}(Q_{11}Q_{22} - Q_{12}Q_{21}) \neq 0$, we have

$$\Gamma^1_{AB} = \Gamma^2_{AB} = 0. \tag{81}$$

The corresponding equations of motion (6) have the following form

$$\partial_\mu J^{\mu,A} + 2\Gamma^A_{B3}J^B_\mu J^{\mu,3} + \Gamma^A_{33}J^3_\mu J^{\mu,3} = 0 \tag{82}$$

$$\partial_\mu J^{\mu,3} = 0. \tag{83}$$

To gain more insight into these equations one should explicitly write the fields $J^{\mu,a}$. First, one needs a suitable realization of the Lie group G. It can be obtained by exponentiation of the elements of the algebra (70)

$$g(x, y, z) = \exp\left(\begin{pmatrix} (1 + b_{21})z & -b_{11}z & x \\ b_{22}z & (1 - b_{12})z & y \\ 0 & 0 & z \end{pmatrix}\right). \tag{84}$$

After a reparametrization we can write a general group element in the form

$$g(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} D(\theta_3) & \begin{matrix}\theta_1 \\ \theta_2\end{matrix} \\ 0\ \ 0 & \exp(\theta_3) \end{pmatrix} \qquad \theta_i \in \mathbf{R} \tag{85}$$

where $D(\theta_3) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \exp\begin{pmatrix} (1+b_{21})\theta_3 & -b_{11}\theta_3 \\ b_{22}\theta_3 & (1-b_{12})\theta_3 \end{pmatrix}$.

Then

$$g^{-1}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} D^{-1}(\theta_3) & \begin{matrix}\exp(-\theta_3)(\det\ D^{-1})(d_{12}\theta_2 - d_{22}\theta_1) \\ \exp(-\theta_3)(\det\ D^{-1})(d_{11}\theta_2 - d_{21}\theta_1)\end{matrix} \\ 0\ \ 0 & \exp(-\theta_3) \end{pmatrix} \tag{86}$$

and

$$\partial_\mu g(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \partial_\mu D(\theta_3) & \begin{matrix}\partial_\mu\theta_1 \\ \partial_\mu\theta_2\end{matrix} \\ 0\ \ 0 & (\partial_\mu\theta_3)\exp(\theta_3) \end{pmatrix}. \tag{87}$$

The fields $J_\mu = g^{-1}\partial_\mu g$ can then be computed

$$J_\mu = \begin{pmatrix} F(\theta_3, \partial_\mu\theta_3) & \begin{matrix}(\det\ D)^{-1}(d_{22}\partial_\mu\theta_1 - d_{12}\partial_\mu\theta_2 + \mathrm{e}^{-z}(\partial_\mu\theta_3)(d_{12}\theta_2 - d_{22}\theta_1)) \\ (\det\ D)^{-1}(d_{21}\partial_\mu\theta_1 - d_{11}\partial_\mu\theta_2 + \mathrm{e}^{-z}(\partial_\mu\theta_3)(d_{11}\theta_2 - d_{21}\theta_1))\end{matrix} \\ 0\ \ 0 & \partial_\mu\theta_3 \end{pmatrix} \tag{88}$$

where $F(\theta_3, \partial_\mu\theta_3) = D^{-1}(\theta_3)\partial_\mu D(\theta_3)$. Reading off the coordinates of the fields J_μ in the basis (t_1, t_2, t_3) one concludes that

(i) $J_\mu^3 = \partial_\mu\theta_3$, i.e. the equation of motion (83) for θ_3 is just the wave equation $\partial_\mu\partial^\mu\theta_3 = 0$ and

(ii) J_μ^1, J_μ^2 are linear in θ_1, θ_2 and their derivatives, i.e. the equations of motion (82) for $\theta_{1,2}$ after substitution of the explicit form of θ_3 turn out to be a system of two coupled linear PDEs for unknown θ_1, θ_2.

Because inverse scattering method is usually not applied to linear PDEs, we do not study this case further.

5.2. *Three–dimensional solvable groups with* $\det B = 0$

The condition $\det B = 0$ allows three possibilities:

(i) three-dimensional nilpotent group, i.e. Heisenberg group,

(ii) centrally extended $Af(1)$ group ($B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$) and

(iii) three-dimensional Abelian group (already considered, see section 3).

5.2.1. *Heisenberg group.*

The Heisenberg group is a nilpotent three-dimensional group. It can be realized as a matrix group of upper triangular 3×3 matrices with unit diagonal. We choose parametrization

$$g(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} 1 & \theta_1 & \theta_3 + \frac{\theta_1\theta_2}{2} \\ 0 & 1 & \theta_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{89}$$

The basis of the corresponding Lie algebra is then

$$t_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad t_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{90}$$

and the nonzero structure coefficients are

$$c_{12}{}^3 = 1 \qquad c_{21}{}^3 = -1. \tag{91}$$

The coordinates of the vector fields J_μ evaluated in the given basis are

$$J_\mu = \left(\partial_\mu \theta_1, \partial_\mu \theta_2, \partial_\mu \theta_3 + \frac{\theta_2}{2} \partial_\mu \theta_1 - \frac{\theta_1}{2} \partial_\mu \theta_2 \right). \tag{92}$$

The differential operators U_a are in this case

$$U_1 = \frac{\partial}{\partial \theta_1} - \frac{\theta_2}{2} \frac{\partial}{\partial \theta_3} \qquad U_2 = \frac{\partial}{\partial \theta_2} + \frac{\theta_1}{2} \frac{\partial}{\partial \theta_3} \qquad U_3 = \frac{\partial}{\partial \theta_3}. \tag{93}$$

Equation (17) is in this case equivalent to a set of equations

$$\begin{aligned} & P_{13} = P_{23} = 0 \qquad Q_{13} = Q_{23} = 0 \\ & P_{11} P_{22} - Q_{11} Q_{22} = Q_{12} Q_{21} - P_{12} P_{21} = P_{33}. \end{aligned} \tag{94}$$

Equation (18) for $a = 1, 2$ is

$$Q_{11} \Gamma^1_{pq} + Q_{12} \Gamma^2_{pq} = 0 \qquad Q_{21} \Gamma^1_{pq} + Q_{22} \Gamma^2_{pq} = 0. \tag{95}$$

Invertibility of Q together with $Q_{13} = Q_{23} = 0$ implies $\det \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \neq 0$ and consequently

$$\Gamma^1_{pq} = \Gamma^2_{pq} = 0. \tag{96}$$

Also it follows that $Q_{1b} \Gamma^b_{3q} = 0$, $Q_{2b} \Gamma^b_{3q} = 0$. Similarly, equation (18) for $a = 3$, $p = 3$ leads to $Q_{3b} \Gamma^b_{3q} = 0$. All these equations imply

$$\Gamma^b_{3q} = 0 \tag{97}$$

and

$$\tilde{\Gamma}_{d3c} \equiv L_{db} \Gamma^b_{3c} = 0. \tag{98}$$

After expressing this equality in coordinates using definition of Γ (7) and some simple algebra, one finds

$$U_3 L_{ij} = 0 \tag{99}$$

$$U_1 L_{33} = 0 \tag{100}$$

$$U_2 L_{33} = 0 \tag{101}$$

$$L_{33} = L_{23,1} - L_{13,2} = U_1 L_{23} - U_2 L_{13}. \tag{102}$$

Similarly, for $b, c \neq 3$ we find using (96)

$$\tilde{\Gamma}_{jbc} \equiv L_{jd} \Gamma^d_{bc} = L_{j3} \Gamma^3_{bc} \tag{103}$$

leading to

$$U_1 L_{11} = L_{13}\Gamma^3_{11} \tag{104}$$

$$L_{13} + U_2 L_{11} = 2L_{13}\Gamma^3_{12} \tag{105}$$

$$2L_{23} + 2U_2 L_{12} - U_1 L_{22} = 2L_{13}\Gamma^3_{22} \tag{106}$$

$$-2L_{13} + 2U_1 L_{12} - U_2 L_{11} = 2L_{23}\Gamma^3_{11} \tag{107}$$

$$-L_{23} + U_1 L_{22} = 2L_{23}\Gamma^3_{12} \tag{108}$$

$$U_2 L_{22} = 2L_{23}\Gamma^3_{22} \tag{109}$$

$$2U_1 L_{13} = 2L_{33}\Gamma^3_{11} \tag{110}$$

$$U_1 L_{23} + U_2 L_{13} = 2L_{33}\Gamma^3_{12} \tag{111}$$

$$2U_2 L_{23} = 2L_{33}\Gamma^3_{22}. \tag{112}$$

Using the last three equations together with (99)–(102) to express Γ^3_{ij} and substituting it into the remaining equations one finds a set of coupled first-order differential equations. Using the fact that nothing depends on θ_3 (see (99), $U_3 = \frac{\partial}{\partial\theta_3}$) one can solve these equations:

$$L_{13} = \alpha\theta_1 + \beta\theta_2 + K_{13} \tag{113}$$

$$L_{23} = \gamma\theta_1 + \delta\theta_2 + K_{23} \tag{114}$$

$$L_{33} = \gamma - \beta \tag{115}$$

$$L_{11} = \frac{L_{13}^2}{L_{33}} + K_{11} \tag{116}$$

$$L_{12} = \frac{L_{13}L_{23}}{L_{33}} + K_{12} \tag{117}$$

$$L_{22} = \frac{L_{23}^2}{L_{33}} + K_{22} \tag{118}$$

where $\alpha, \beta, \gamma, \delta, K_{i,j} \in \mathbf{R}, \beta \neq \gamma$ are parameters such that $K_{11}K_{22} - K_{12}^2 \neq 0$. The corresponding equations of motion are of the following form

$$\frac{\partial^2\theta_1}{\partial x_0^2} - \frac{\partial^2\theta_1}{\partial x_1^2} = 0 \tag{119}$$

$$\frac{\partial^2\theta_2}{\partial x_0^2} - \frac{\partial^2\theta_2}{\partial x_1^2} = 0 \tag{120}$$

$$\frac{\partial^2\theta_3}{\partial x_0^2} - \frac{\partial^2\theta_3}{\partial x_1^2} + F(\theta_1, \theta_2) = 0 \tag{121}$$

where $F(\theta_1, \theta_2)$ is a certain function of θ_1, θ_2 and their derivatives. It is clear that the homogeneous wave equations (119) and (120) can be solved explicitly and then (121) is an inhomogeneous wave equation. A corresponding Lax pair can be found by solving equations (18)–(21). To sum up, the only possible generalized principal chiral model for the Heisenberg group expressible by the Lax operators (13) and (14) is again equivalent to the inhomogeneous wave equation.

5.2.2. Centrally extended $Af(1)$ group. We consider a 2×2 matrix realization of this group with the parametrization

$$g(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} \exp\left(\frac{\theta_1 - \theta_3}{2}\right) & \exp\left(\frac{\theta_1}{2}\right)\frac{\theta_2}{2} \\ 0 & \exp\left(\frac{\theta_1 + \theta_3}{2}\right) \end{pmatrix}. \tag{122}$$

In the corresponding Lie algebra we choose a basis

$$t_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \qquad t_1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{123}$$

The nonzero structure coefficients for this choice of basis are

$$c_{23}{}^2 = 1 \qquad c_{32}{}^2 = -1. \tag{124}$$

The coordinates of the vector fields J_μ in this basis are

$$J_\mu = \left(\partial_\mu\theta_1, \mathrm{e}^{-\frac{\theta_3}{2}}\left(\partial_\mu\theta_2 - \frac{1}{2}\theta_2\partial_\mu\theta_3\right), \partial_\mu\theta_3\right). \tag{125}$$

The differential operators U_a are in this case

$$U_1 = \frac{\partial}{\partial\theta_1} \qquad U_2 = \mathrm{e}^{(-\theta_3/2)}\frac{\partial}{\partial\theta_2} \qquad U_3 = \frac{1}{2}\theta_2\frac{\partial}{\partial\theta_2} + \frac{\partial}{\partial\theta_3}. \tag{126}$$

We have used an approach rather similar to the one used in the case of $Af(1)$. First we have used equation (18) with $a = 1, 3$ to derive a linear relation between Γ^i_{jk}:

$$0 = Q_{1b}\Gamma^b_{pq} \qquad 0 = Q_{3b}\Gamma^b_{pq} \tag{127}$$

reducing (together with the invertibility of Q) the possible values of Γ to

$$\Gamma^i_{pq} = \kappa^i \Delta_{pq} \tag{128}$$

where κ^i, $\Delta_{pq} = \Delta_{qp} \in \mathbf{R}$.

In the next step we put the above given expressions for Γ into the definition of the connection (7) and solve it with respect to derivatives of the metric L.

Using those PDEs for L we calculate a necessary condition for the existence of a Lax pair. We evaluate the difference of second derivatives

$$\frac{\partial}{\partial\theta_a}\frac{\partial}{\partial\theta_b}L_{ij} - \frac{\partial}{\partial\theta_b}\frac{\partial}{\partial\theta_a}L_{ij} \tag{129}$$

in terms of L_{kl}. Since this difference must be zero, we obtain a set of 18 (for $i, j \in \{1, 2, 3\}$ and $(a, b) = (1, 2), (1, 3), (2, 3)$) equations for six components of the metric L_{kl} and nine constants κ^j, Δ_{pq}.

We have solved equations (129) using Maple V computer algebra system only, neither Mathematica 4 nor Reduce 3.6 were able to solve it. Therefore we have to rely on the results of Maple and we are not able to independently check the completeness of the solution.

All possible connections allowing invertible metric L and Lax formulation of equations of motion (13)–(15) appear to be of one of the following two forms (we recall that $J^2_\mu = \mathrm{e}^{-\frac{\theta_3}{2}}(\partial_\mu\theta_2 - \frac{1}{2}\theta_2\partial_\mu\theta_3)$):

(i) $\Gamma^1_{pq} = \Gamma^3_{pq} = 0, \Gamma^2_{1q} = \Gamma^2_{22} = 0, \Gamma^2_{23} = -\frac{1}{2}, \Gamma^2_{33} \in \mathbf{R}$. The corresponding equations of motion are

$$\partial_\mu\partial^\mu\theta_1 = 0 \qquad \partial_\mu J^{\mu,2} - \frac{1}{2}J^{\mu,2}\,\partial_\mu\theta_3 + \Gamma^2_{33}\partial_\mu\theta_3\partial^\mu\theta_3 = 0 \qquad \partial_\mu\partial^\mu\theta_3 = 0 \tag{130}$$

i.e., the equations of motion for θ_1, θ_3 are the wave equations and can be solved explicitly. The equation of motion for θ_2 is linear after substitution of the solution θ_3 because $J^{\mu,2}$ is linear in θ_2.

(ii) $\Gamma^3_{pq} = \Gamma^2_{pq} = 0, \Gamma^1_{1j} = 0, \Gamma^1_{22}, \Gamma^1_{23}, \Gamma^1_{33} \in \mathbf{R}$. The equations of motion are

$$\partial_\mu\partial^\mu\theta_1 + F(\theta_2, \theta_3) = 0 \qquad \partial_\mu J^{\mu,2} = 0 \qquad \partial_\mu\partial^\mu\theta_3 = 0 \tag{131}$$

where $F(\theta_2, \theta_3)$ is a certain given function of θ_2, θ_3 and their derivatives. In this case we have again the wave equation for θ_3. After substituting the solution of this equation into $\partial_\mu J^{\mu,2} = 0$ we have a linear PDE for θ_2 and finally substituting both θ_2, θ_3 into an equation of motion for θ_1 we have an inhomogeneous wave equation for θ_1:

$$\partial_\mu\partial^\mu\theta_1 + F(\theta_2, \theta_3) = 0.$$

This case also includes the model with $\Gamma^a_{pq} = 0$.

To conclude, in the case of centrally extended group $Af(1)$ we have found no intrinsically nonlinear model but one should be aware that the completeness of this result relies on the computation done only in one computer algebra system.

6. Conclusions

We have analysed generalized principal chiral models given by the action of the form (1) where the target manifold of the fields are two- and three-dimensional connected and simply connected non-semisimple Lie groups. We have found that in these cases all but one equations of motions admitting Lax formulation (13)–(15) can be brought to linear PDEs.

The only truly nonlinear system of equations comes from the generalized principal model on the two-dimensional solvable group with the non-constant metric

$$L(\theta_1, \theta_2) = \begin{pmatrix} \frac{-1+K^2\kappa^2}{\kappa^2}\alpha e^{\theta_1} & -K\alpha e^{\theta_1} \\ -K\alpha e^{\theta_1} & \alpha e^{\theta_1} \end{pmatrix} \tag{132}$$

where $K \in \mathbf{R}, \alpha, \kappa \in \mathbf{R}\backslash\{0\}$. Its equations of motion read

$$\partial_\nu\partial^\nu\theta_1 + \frac{1}{2}\partial_\nu\theta_1\partial^\nu\theta_1 - \frac{1}{2}\kappa^2\left(K^2\partial_\nu\theta_1\partial^\nu\theta_1 - 2K e^{-\theta_1}\partial_\nu\theta_1\partial^\nu\theta_2 + e^{-2\theta_1}\partial_\nu\theta_2\partial^\nu\theta_2\right) = 0 \tag{133}$$

$$\partial_\nu\partial^\nu\theta_2 - K e^{\theta_1}\partial_\nu\partial^\nu\theta_1 = \partial_\nu\theta_1\partial^\nu\theta_2 \tag{134}$$

and the Lax pair is

$$X_0 = \begin{pmatrix} \partial_0 + \lambda + \frac{1}{2}\left(J^1_0 + K\kappa J^1_1 - \kappa J^2_1\right), & -2\lambda + J^1_1 - \kappa J^2_0 + K\kappa J^1_0 \\ 0 & \partial_0 \end{pmatrix} \tag{135}$$

$$X_1 = \begin{pmatrix} \partial_1 + \lambda + \frac{1}{2}\left(J^1_1 + K\kappa J^1_0 - \kappa J^2_0\right), & -2\lambda + J^1_0 - \kappa J^2_1 + K\kappa J^1_1 \\ 0 & \partial_1 \end{pmatrix} \tag{136}$$

where $J^1_\mu = \partial_\mu\theta_1$, $J^2_\mu = e^{-\theta_1}\partial_\mu\theta_2$ and λ is a free parameter that can be used for the inverse scattering method.

References

[1] Zakharov V E and Mikhailov A V 1978 Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique *J. Exp. Theor. Phys.* **74** 1953 (in Russian)

[2] Sochen N 1997 Integrable generalized principal chiral models *Phys. Lett.* B **391** 374

[3] Cherednik I V 1981 On the integrability of the equation for the two-dimensional chiral asymmetric O(3) field and its quantum analog *Jadernaja fizika* **33** 278 (in Russian)

[4] Holod P I 1987 Hidden symmetry of the Landau–Lifshitz equation, its higher analogues and dual equation for asymmetric chiral field *Theor. Math. Phys.* **70** 18 (in Russian)

[5] Hlavatý L 2000 Principal models on a solvable group with nonconstant metric *Phys. Lett.* A **275** 419

[6] Takhtadzhyan L A and Faddeev L D 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)

[7] Bryant L 1995 Introduction to Lie groups and symplectic geometry *Geometry and Quantum Field Theory* ed D S Freed and K K Uhlenbeck (Chicago, IL: AMS) pp 5–182